Problem 1.53

(a) Which of the vectors in Problem 1.15 can be expressed as the gradient of a scalar? Find a scalar function that does the job.

(b) Which can be expressed as the curl of a vector? Find such a vector.

Solution

The three vector fields in Problem 1.15 are as follows.

$$\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$$

$$\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$$

$$\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$$

Calculate the divergence of each function.

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = (2x) + (0) + (-2x) = 0$$

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx) = (y) + (2z) + (3x) = y + 2z + 3x$$

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz) = (0) + (2x) + (2y) = 2x + 2y$$

Since the divergence of $\mathbf{v}_a$ is zero, there exists a vector potential function $\mathbf{A}_a$ such that $\mathbf{v}_a = \nabla \times \mathbf{A}_a$.

$$x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

For simplicity, choose

$$\mathbf{A}_a = 0\hat{\mathbf{x}} - x^2z\hat{\mathbf{y}} - \frac{3}{2}x^2z^2\hat{\mathbf{z}}.$$

Calculate the curl of each function.

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0)$$

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x)$$

$$\nabla \times \mathbf{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} = \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) = \mathbf{0}$$

Since the curl of $\mathbf{v}_c$ is zero, there exists a scalar potential function $-\phi_c$ such that $\mathbf{v}_c = -\nabla\phi_c$.

$$\langle y^2, 2xy + z^2, 2yz\rangle = -\left\langle \frac{\partial\phi_c}{\partial x}, \frac{\partial\phi_c}{\partial y}, \frac{\partial\phi_c}{\partial z} \right\rangle$$

For simplicity, choose $\phi_c(x, y, z) = -xy^2 - yz^2$.